A note on the δ-singularities of the static electric and magnetic fields

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Abstract. The $\delta-$singularities of the electric and magnetic fields in the static case are established based on the regularized $\delta_\varepsilon(\boldsymbol{r})$ function introduced by Jackson [1].

1. Introduction

The problem of regularization when searching the $\delta-$singularities of the multipole electromagnetic fields is treated in the present paper in a simple procedure. In the static case, we can use a generalization of a simple identity implying a regularized derivative of $1/r$ and the Jackson regularized function $\delta_\varepsilon(\boldsymbol{r})$ [1] and [2] - equation (2).
In Section 2 we treat the electrostatic field, while in section 3, the magnetic field. In both the cases, we discuss the singularities for the first three multipoles: dipole, quadrupole and octopole. In Section 4, as a simple mathematical digression, we generalize the results for arbitrary multipole orders.

2. Electrostatic field

In the present section we search the $\delta-$singularities of the electrostatic fields of the electric dipole, quadrupole and octopole. These fields correspond to the following multipole expansion written here in the case of an electric neutral charge distribution in a finite space region $\mathcal{D}$ [1], [3]:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\boldsymbol{e}_i\left(p_j\partial_i\partial_j\frac{1}{r} - \frac{1}{2}\mathsf{P}_{jk}\partial_i\partial_j\partial_k\frac{1}{r} + \frac{1}{6}\mathsf{P}_{jkl}\partial_i\partial_j\partial_k\partial_l\frac{1}{r} + \dots\right) . \quad (1)$$

p_i, P_{ij}, P_{ijk} are the Cartesian components of the electric multipole moments:

$$p_i = \int_{\mathcal{D}} \mathrm{d}^3x\, x_i\rho(\boldsymbol{r}),\quad \mathsf{P}_{ij} = \int_{\mathcal{D}} \mathrm{d}^3x\, x_ix_j\,\rho(\boldsymbol{r}),\quad \mathsf{P}_{ijk} = \int_{\mathcal{D}} \mathrm{d}^3x\, x_ix_jx_k\,\rho(\boldsymbol{r}) .$$

The origin O of the Cartesian axes is chosen in the domain $\mathcal{D}$ and $\boldsymbol{e}_i,\ i = 1, 2, 3$ are the unit vectors of these axes.

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The various terms from the multipole expansion (1), considered as the fields corresponding to point-like multipoles, are defined as functions on $\mathbb{R}^3$ having as support the entire space without the point O. The extensions of such functions to the entire space can be realized as distributions (generalized functions) adding some distributions with point-like support to the expressions defined only for $r \neq 0$. These last distributions are generally linear combinations of the $\delta-$functions and their derivatives. These $\delta-$type distributions correspond, actually, to the extension of the various orders partial derivatives of $1/r$.

The well-known case of the electric dipole of the moment $\boldsymbol{p}$ is treated employing the extension as distribution of the second order derivative $\partial_i\partial_j(1/r)$ [4]. In the present paper we make use of the regularized $\delta-$function $\delta_\varepsilon(\boldsymbol{r}),\ \varepsilon \to +0$, introduced in Jackson's book [1]:

$$\delta_\varepsilon(\boldsymbol{r}) = -\frac{1}{4\pi}\Delta\frac{1}{\sqrt{r^2+\varepsilon^2}} = \frac{1}{4\pi}\frac{3\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}} :$$

$$\langle\delta_\varepsilon(\boldsymbol{r}),\, \phi(\boldsymbol{r})\rangle = \int \mathrm{d}^3x\,\delta_\varepsilon(\boldsymbol{r})\phi(\boldsymbol{r}) \stackrel{\varepsilon\to 0}{\longrightarrow} \phi(0) = \langle\,\delta,\, \phi\rangle\ . \tag{2}$$

The test function $\phi(\boldsymbol{r})$ is an arbitrary element of the space of the Dirac function δ which can be considered as the subspace of continuous and differentiable functions from $\mathcal{L}^2$.

The second order derivative $\partial_i\partial_j(1/r)$ can be treated by a procedure of isolating the $\delta-$singularities suggested by equation (2) from Ref. [2], writing the regularized distribution

$$\partial_i\partial_j\frac{1}{\sqrt{r^2+\varepsilon^2}} = \frac{3x_ix_j}{(r^2+\varepsilon^2)^{5/2}} - \frac{\delta_{ij}}{(r^2+\varepsilon^2)^{3/2}}$$
$$= \frac{3x_ix_j - r^2\delta_{ij}}{(r^2+\varepsilon^2)^{5/2}} - \frac{\varepsilon^2\delta_{ij}}{(r^2+\varepsilon^2)^{5/2}} = \frac{3x_ix_j - r^2\delta_{ij}}{(r^2+\varepsilon^2)^{5/2}} - \frac{4\pi}{3}\delta_{ij}\delta_\varepsilon(\boldsymbol{r})\ . \tag{3}$$

Denoting $(D)_{(0)}$ a distribution with point-like support, we can write in the present case

$$\left(\partial_i\partial_j\frac{1}{r}\right)_{(0)} = -\frac{4\pi}{3}\delta_{ij}\delta(\boldsymbol{r})\ . \tag{4}$$

This result, introduced in the expression of the electric point-like dipole field, becomes

$$\left(\boldsymbol{E}^{(1)}\right)_{(0)} = -\frac{1}{3\varepsilon_0}\,\boldsymbol{p}\,\delta(\boldsymbol{r})\ . \tag{5}$$

Let be the regularized expression of the field $\boldsymbol{E}^{(2)}$ of the point-like quadrupole:

$$\left(\boldsymbol{E}^{(2)}(\boldsymbol{r})\right)_{reg} = -\frac{1}{8\pi\varepsilon_0}\,\boldsymbol{e}_i\mathsf{P}_{jk}\partial_i\partial_j\partial_k\frac{1}{r_\varepsilon}\ , \tag{6}$$

where, for simplifying the notation, it is introduced

$$r_\varepsilon = \sqrt{r^2+\varepsilon^2}\ . \tag{7}$$

A totally symmetric $n-th$ order tensor can be projected on the subspace of the totally symmetric and trace free (**STF**) tensors. In the particular case $n = 2$, this projection is realized writing the decomposition

$$\mathsf{P}_{ij} = \mathcal{P}_{ij} + \Lambda\,\delta_{ij} \tag{8}$$

and choosing the parameter Λ such that $\mathcal{P}_{ii}=0$, i.e.

$$\Lambda = \frac{1}{3}\mathsf{P}_{ll} \ .$$

Inserting equation (8) in the regularized expression (6), we can write

$$\left(\boldsymbol{E}^{(2)}(\boldsymbol{r})\right)_{reg} = -\frac{1}{8\pi\varepsilon_0}\boldsymbol{e}_i\,\mathcal{P}_{jk}\partial_i\partial_j\partial_k\frac{1}{r_\varepsilon} + \frac{1}{2\varepsilon_0}\Lambda\,\boldsymbol{e}_i\partial_i\delta_\varepsilon(\boldsymbol{r}) \ .$$

One of the singularities of the field is generated by the last term from the previous equation. We have to search the singularities of the contraction of the **STF** tensor $\mathcal{P}_{ij}$ with the derivative tensor. The regularized expression of the third-order derivative is given by

$$\partial_i\partial_j\partial_k\frac{1}{r_\varepsilon} = -\frac{15\,x_ix_jx_k}{r_\varepsilon^7} + \frac{3(x_i\delta_{jk}+x_j\delta_{ik}+x_k\delta_{ij})}{r_\varepsilon^5} = -\frac{15\,x_ix_jx_k}{r_\varepsilon^7} + \frac{3\,\delta_{\{ij}\,x_{k\}}}{r_\varepsilon^5} \ .$$

The notation $\{i_1\,\ldots\,i_n\}$ symbolizes the sum over all the transpositions of the indexes $i_1\ldots i_n$ which correspond to distinct terms. Applying the same procedure as in the case of equation (3),

$$\partial_i\partial_j\partial_k\frac{1}{r_\varepsilon} = -\frac{15\,x_ix_jx_k}{r_\varepsilon^7} + \frac{3\,\delta_{\{ij}x_{k\}}}{r_\varepsilon^5} = \frac{-15\,x_ix_jx_k + 3r^2\delta_{\{ij}x_{k\}}}{r_\varepsilon^7} + \frac{3\varepsilon^2\,\delta_{\{ij}x_{k\}}}{r_\varepsilon^7} \ . \tag{9}$$

Writing the partial derivative of $\delta_\varepsilon(\boldsymbol{r})$, we obtain

$$\frac{\varepsilon^2\,x_i}{r_\varepsilon^7} = -\frac{4\pi}{15}\,\partial_i\delta_\varepsilon(\boldsymbol{r}) \ ,$$

which, inserted in equation (9), gives

$$\partial_i\partial_j\partial_k\frac{1}{r_\varepsilon} = \frac{-15\,x_ix_jx_k + 3r^2\delta_{\{ij}x_{k\}}}{r_\varepsilon^7} - \frac{4\pi}{5}\delta_{\{ij}\partial_{k\}}\delta_\varepsilon(\boldsymbol{r}) \ , \tag{10}$$

such that the $\delta-$singularities of the third-order derivative are given by

$$\left(\partial_i\partial_j\partial_k\frac{1}{r}\right)_{(0)} = -\frac{4\pi}{5}\delta_{\{ij}\partial_{k\}}\delta(\boldsymbol{r}) \ , \tag{11}$$

a well-known result [4]. Therefore, the singularity of the contraction of the **STF** moment with the derivative tensor is given by

$$\left(\mathcal{P}_{jk}\partial_i\partial_j\partial_k\frac{1}{r}\right)_{(0)} = -\frac{8\pi}{5}\mathcal{P}_{ij}\partial_j\delta(\boldsymbol{r}) \ .$$

Finally, for the point-like quadrupole, we obtain for the electric field

$$\left(\boldsymbol{E}^{(2)}(\boldsymbol{r})\right)_{(0)} = \boldsymbol{e}_i\left[\frac{1}{5\varepsilon_0}\,\mathcal{P}_{ij}\partial_j\delta(\boldsymbol{r}) + \frac{1}{2\varepsilon_0}\Lambda\,\partial_i\delta(\boldsymbol{r})\right] \ ,$$

or, using a tensorial notation,

$$\left(\boldsymbol{E}^{(2)}(\boldsymbol{r})\right)_{(0)} = \frac{1}{5\varepsilon_0}\boldsymbol{\mathcal{P}}^{(2)}||\boldsymbol{\nabla}\delta(\boldsymbol{r}) + \frac{1}{2\varepsilon_0}\Lambda\boldsymbol{\nabla}\delta(\boldsymbol{r}) \ . \tag{12}$$

In the last equation, it is employed the general notation $\mathsf{T}^{(n)}$ for the $n-th$ order tensor and, also, the notation for the tensor contraction:

$$(\mathsf{A}^{(n)}||\mathsf{B}^{(m)})_{i_1\cdots i_{|n-m|}} = \begin{cases} A_{i_1\cdots i_{n-m}j_1\cdots j_m}B_{j_1\cdots j_m} & ,\ n>m \\ A_{j_1\cdots j_n}B_{j_1\cdots j_n} & ,\ n=m \\ A_{j_1\cdots j_n}B_{j_1\cdots j_n i_1\cdots i_{m-n}} & ,\ n<m \end{cases} \ .$$

Searching directly the singularities associated to distributions defined as contractions of electric (or magnetic) moments and derivative tensors represents the basic procedure adopted in the present paper. This procedure implies an appreciable simplicity of the calculation especially for the higher-order multipoles.
We point out the invariance of the electrostatic field for $\left(\boldsymbol{E}^{(2)}(\boldsymbol{r})\right)_{r\neq 0}$ to the substitution of the "primitive " moment $\mathsf{P}^{(2)}$ by the **STF** one, $\mathsf{P}^{(2)} \to \boldsymbol{\mathcal{P}}^{(2)}$, such that

$$\left(\boldsymbol{E}^{(2)}(\boldsymbol{r})\right)_{r\neq 0} = -\frac{1}{8\pi\varepsilon_0}\boldsymbol{\mathcal{P}}^{(2)}||\boldsymbol{\nabla}^3\frac{1}{r} \ . \tag{13}$$

Obviously, searching the $\delta-$singularities of $\boldsymbol{E}^{(2)}$ starting from this last expression, the singular term containing the parameter Λ from equation (12) is lost. The primitive tensor $\mathsf{P}^{(2)}$ includes all the necessary elements for finding the $\delta-$singularities of the field. Employing, for example, the multipole expansion in terms of the spherical functions $Y_{lm}(\theta,\varphi)$, it is equivalent to employing the expansion (13) yielding therefore, a wrong result for such type of singularities.
Let us consider the regularized expression of the electric field $\boldsymbol{E}^{(3)}(\boldsymbol{r})$:

$$\left(\boldsymbol{E}^{(3)}\right)_{reg} = \frac{1}{24\pi\varepsilon_0}\mathsf{P}^{(3)}||\boldsymbol{\nabla}^4\frac{1}{r_\varepsilon} = \frac{1}{24\pi\varepsilon_0}\boldsymbol{e}_i\,\mathsf{P}_{jkl}\,\partial_i\partial_j\partial_k\partial_l\frac{1}{r_\varepsilon} \ . \tag{14}$$

The **STF** projection of the tensor $\mathsf{P}^{(3)}$ is introduced by the following decomposition:

$$\mathsf{P}_{ijk} = \mathcal{P}_{ijk} + \Lambda_{\{i}\delta_{jk\}} \ . \tag{15}$$

The parameters Λ_i are established requiring the vanishing of all the traces of the symmetric tensor $\mathsf{P}^{(3)}$:

$$\mathsf{P}_{ijj} = 0,\ (i = 1,\,2,\,3) \ .$$

The results are given by

$$\Lambda_i = \frac{1}{5}\,\mathsf{P}_{ijj} \ . \tag{16}$$

The insertion of equation (15) in equation (14) gives

$$\begin{aligned}\left(\boldsymbol{E}^{(3)}\right)_{reg} &= \frac{1}{24\pi\varepsilon_0}\left[\boldsymbol{\mathcal{P}}^{(3)}||\boldsymbol{\nabla}^4\frac{1}{r_\varepsilon} + \frac{3}{24\pi\varepsilon_0}\boldsymbol{e}_i\Lambda_j\partial_i\partial_j\Delta\frac{1}{r_\varepsilon}\right] \\ &= \frac{1}{24\pi\varepsilon_0}\boldsymbol{\mathcal{P}}^{(3)}||\boldsymbol{\nabla}^4\frac{1}{r_\varepsilon} - \frac{1}{2\varepsilon_0}\boldsymbol{e}_i\Lambda_j\partial_j\partial_i\delta_\varepsilon(\boldsymbol{r}) \ .\end{aligned} \tag{17}$$

The fourth-order derivative of $1/r$ is given by

$$\partial_i\partial_j\partial_k\partial_l\frac{1}{r} = \frac{7!!\,x_ix_jx_kx_l}{r^9} - \frac{5!!\,\delta_{\{ij}\,x_kx_{l\}}}{r^7} + \frac{3\,\delta_{\{ij}\,\delta_{kl\}}}{r^5} \ .$$

We are interested only in writing the contraction $\mathcal{P}_{jkl}\partial_i\partial_j\partial_k\partial_l(1/r)$ and:

$$\mathcal{P}_{jkl}\partial_i\partial_j\partial_k\partial_l\frac{1}{r} = \mathcal{P}_{jkl}\left(\frac{7!!\,x_ix_jx_kx_l}{r^9} - \frac{5!!\,\delta_{\{ij}\,x_kx_{l\}}}{r^7}\right) \ ,$$

since $\mathcal{P}_{jkl}\delta_{\{ij}\,\delta_{kl\}} = 0$. It is easy to see that the same formula applies to $1/r_\varepsilon$ by the simple substitution $r \to r_\varepsilon$. Writing the corresponding regularized expression and applying the

same procedure as in the case of the third-order derivative for obtaining equation (10), we write

$$\mathcal{P}_{jkl}\partial_i\partial_j\partial_k\partial_l\frac{1}{r_\varepsilon} = \mathcal{P}_{jkl}\left(\frac{7!!\,x_ix_jx_kx_l - 5!!\,r^2\,\delta_{\{ij}\,x_kx_{l\}}}{r_\varepsilon^9} - 15\frac{\varepsilon^2\,\delta_{\{ij}\,x_kx_{l\}}}{r_\varepsilon^9}\right) \ . \tag{18}$$

Expressing the second partial derivative of $\delta_\varepsilon(\boldsymbol{r})$, one obtains

$$\frac{\varepsilon^2\,x_ix_j}{r_\varepsilon^9} = \frac{4\pi}{7!!}\partial_i\partial_j\,\delta_\varepsilon(\boldsymbol{r}) + \frac{\varepsilon^2\,\delta_{ij}}{7\,r_\varepsilon^7} \ .$$

The insertion of this equation in the last fraction from equation (18) gives

$$\mathcal{P}_{jkl}\partial_i\partial_j\partial_k\partial_l\frac{1}{r_\varepsilon} = \mathcal{P}_{jkl}\left(\frac{7!!\,x_ix_jx_kx_l - 5!!\,r^2\,\delta_{\{ij}\,x_kx_{l\}}}{r_\varepsilon^9} - \frac{4\pi}{7}\delta_{\{ij}\,\partial_k\partial_{l\}}\delta_\varepsilon(\boldsymbol{r})\right) \ .$$

From the last result, we identify the $\delta-$singularity:

$$\begin{aligned}\left(\mathcal{P}_{jkl}\,\partial_i\partial_j\partial_k\partial_l\frac{1}{r}\right)_{(0)} &= -\frac{4\pi}{7}\mathcal{P}_{jkl}\left(\delta_{ij}\partial_k\partial_l\,\delta(\boldsymbol{r}) + \delta_{ik}\partial_j\partial_l\,\delta(\boldsymbol{r}) + \delta_{il}\partial_j\partial_k\,\delta(\boldsymbol{r})\right)\\ &= -4\pi\,\frac{3}{7}\mathcal{P}_{ijk}\partial_j\partial_k\,\delta(\boldsymbol{r}) \ ,\end{aligned} \tag{19}$$

where the vanishing of the contraction of $\mathcal{P}_{jkl}$ with δ_{jk}, δ_{jl}, δ_{kl} and the symmetry properties of the $\boldsymbol{\mathcal{P}}^{(3)}$ components are considered. With tensorial notation,

$$\left(\boldsymbol{\mathcal{P}}^{(3)}||\boldsymbol{\nabla}^4\frac{1}{r}\right)_{(0)} = -4\pi\,\frac{3}{7}\boldsymbol{\mathcal{P}}^{(3)}||\boldsymbol{\nabla}^2\delta(\boldsymbol{r}) \ . \tag{20}$$

This last result together with equation (17) and the limit for $\varepsilon \to 0$ in the distribution space leads to the $\delta-$singularity of $\boldsymbol{E}^{(3)}$:

$$\left(\boldsymbol{E}^{(3)}(\boldsymbol{r})\right)_{(0)} = -\frac{1}{14\,\varepsilon_0}\,\boldsymbol{\mathcal{P}}^{(3)}||\boldsymbol{\nabla}^2\delta(\boldsymbol{r}) - \frac{1}{2\,\varepsilon_0}\boldsymbol{\Lambda}||\boldsymbol{\nabla}^2\delta(\boldsymbol{r}) \ , \tag{21}$$

where $\boldsymbol{\Lambda} = \Lambda_i\boldsymbol{e}_i$.

This procedure can be generalized to any arbitrary higher-order n.

3. Magnetostatic field

We consider the first three terms from the magnetostatic field expansions [1], [3]:

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}&\left(\boldsymbol{\nabla}^2||\frac{\boldsymbol{m}}{r} - \Delta\frac{\boldsymbol{m}}{r} - \frac{1}{2}\boldsymbol{\nabla}^3||\frac{\mathsf{M}^{(2)}}{r} + \frac{1}{2}\boldsymbol{\nabla}||\Delta\frac{\mathsf{M}^{(2)}}{r} + \frac{1}{6}\boldsymbol{\nabla}^4||\frac{\mathsf{M}^{(3)}}{r}\right.\\ &\left.-\frac{1}{6}\boldsymbol{\nabla}^2||\Delta\frac{\mathsf{M}^{(3)}}{r} + \ldots\right) = \frac{\mu_0}{4\pi}\boldsymbol{e}_i\left(\partial_i\partial_j\frac{m_j}{r} - \Delta\frac{m_i}{r} - \frac{1}{2}\partial_i\partial_j\partial_k\frac{\mathsf{M}_{jk}}{r}\right.\\ &\left.+\frac{1}{2}\partial_j\Delta\frac{\mathsf{M}_{ji}}{r} + \frac{1}{6}\partial_i\partial_j\partial_k\partial_l\frac{\mathsf{M}_{jkl}}{r} - \frac{1}{6}\partial_j\partial_k\Delta\frac{\mathsf{M}_{jki}}{r} + \ldots\right) \ .\end{aligned} \tag{22}$$

The magnetic moments are defined by [1], [3]

$$\begin{aligned}&m_i = \frac{1}{2}\int_{\mathcal{D}}\mathrm{d}^3x\,\left(\boldsymbol{r}\times\boldsymbol{J}\right)_i \ , \quad \mathsf{M}_{ik} = \frac{2}{3}\int_{\mathcal{D}}\mathrm{d}^3x\,x_i\,\left(\boldsymbol{r}\times\boldsymbol{J}\right)_k \ ,\\ &\mathsf{M}_{ijk} = \frac{3}{4}\int_{\mathcal{D}}\mathrm{d}^3x\,x_ix_j\left(\boldsymbol{r}\times\boldsymbol{J}\right)_k \ .\end{aligned} \tag{23}$$

Let us write the regularized expression of the dipolar magnetic field:

$$(\boldsymbol{B}(\boldsymbol{r}))_{reg} = \frac{\mu_0}{4\pi}\left(\boldsymbol{e}_i\partial_i\partial_j\frac{m_j}{r} + 4\pi\boldsymbol{m}\,\delta_\varepsilon(\boldsymbol{r})\right) .$$

Inserting equations (3) and (4), we obtain the following result for the singular term

$$(\boldsymbol{B}(\boldsymbol{r}))_{(0)} = -\frac{\mu_0}{3}\boldsymbol{m}\,\delta(\boldsymbol{r}) + \mu_0\,\boldsymbol{m}\,\delta(\boldsymbol{r}) = \frac{2\mu_0}{3}\,\boldsymbol{m}\,\delta(\boldsymbol{r}) . \tag{24}$$

Comparing equations (5) and (24), it is seen the difference between the two expressions concerning the proportional factor associated to the dipolar moment $\boldsymbol{p}$ or $\boldsymbol{m}$. Formally, this difference is due to the supplementary term $\Delta(\boldsymbol{m}/r)$ in the magnetic case. Physically, the difference becomes easily obvious if one considers the fictitious magnetic shells (or sheets) employed in the Ampère formalism. It suffices to consider the case of the point-like magnetic dipole which can be taken as the limit of a current loop of infinitesimal size. For finite dimensions, the field of the loop can be derived from a scalar potential Φ_m which is defined by an integral on the corresponding sheet and having a "jump" in all their points. Just this jump generates the δ-singularity corresponding to the second term from equation (24). An explicit calculation of this limit when the loop concentrates in a point is given in Ref. [5].

Concerning the higher-order magnetic multipoles, it is necessary to specify the procedure of projecting the corresponding moments on the **STF** tensor subspace. In the case of the quadrupolar magnetic moment, the components M_{ij} are defined in equation (23). These components, not being symmetric in the two indexes, will involve two steps in establishing the **STF** tensor $\boldsymbol{\mathcal{M}}^{(2)} = \boldsymbol{\mathcal{T}}(\mathbf{M}^{(2)})$, where by $\boldsymbol{\mathcal{T}}$ is denoted the correspondence between an arbitrary tensor and its **STF** projection. We establish firstly the symmetric projection denoted here by $\overset{\leftrightarrow}{\mathbf{M}}^{(2)}$ and, secondly, we apply the known procedure of establishing the **STF** projection of this one. Let us write the identity

$$\mathsf{M}_{ij} = \frac{1}{2}\left(\mathsf{M}_{ij} + \mathsf{M}_{ji}\right) + \frac{1}{2}\left(\mathsf{M}_{ij} - \mathsf{M}_{ji}\right) = \overset{\leftrightarrow}{\mathsf{M}}_{ij} + \frac{1}{2}\left(\mathsf{M}_{ij} - \mathsf{M}_{ji}\right) . \tag{25}$$

In this case ($n = 2$), $\boldsymbol{\mathcal{M}}^{(2)} = \overset{\leftrightarrow}{\mathbf{M}}^{(2)}$ since $\mathsf{M}_{jj} = 0$ and, consequently, corresponds to the **STF** projection. Therefore,

$$\mathsf{M}_{ij} = \mathcal{M}_{ij} + \frac{1}{2}\left(\mathsf{M}_{ij} - \mathsf{M}_{ji}\right) . \tag{26}$$

Let be the regularized expression of the 4-polar magnetic field:

$$\left(\boldsymbol{B}^{(2)}(\boldsymbol{r})\right)_{reg} = \frac{\mu_0}{8\pi}\boldsymbol{e}_i\left[-\mathsf{M}_{jk}\partial_i\partial_j\partial_k\frac{1}{r_\varepsilon} + \mathsf{M}_{ji}\partial_j\Delta\frac{1}{r_\varepsilon}\right]$$

and inserting equations (2) and (26),

$$\left(\boldsymbol{B}^{(2)}(\boldsymbol{r})\right)_{reg} = -\frac{\mu_0}{8\pi}\boldsymbol{e}_i\left[\mathcal{M}_{jk}\,\partial_i\partial_j\partial_k\frac{1}{r_\varepsilon} + 4\pi\mathsf{M}_{ji}\partial_j\delta_\varepsilon(\boldsymbol{r})\right] ,$$

where the vanishing of the contraction $\left(\mathsf{M}_{jk} - \mathsf{M}_{kj}\right)\partial_i\partial_j\partial_k$ is considered. It remains to separate the part generating the $\delta-$singularity of the expression $\mathcal{M}_{jk}\,\partial_i\partial_j\partial_k(1/r_\varepsilon)$ from

the last equation. This objective is performed in the same manner as in the case of the electric field $\boldsymbol{E}^{(2)}$, employing equation (11). We obtain the final result

$$\left(\boldsymbol{B}^{(2)}(\boldsymbol{r})\right)_{(0)} = \mu_0 \left(\frac{1}{5}\boldsymbol{\mathcal{M}}^{(2)}||\boldsymbol{\nabla}\delta(\boldsymbol{r}) - \frac{1}{2}\boldsymbol{\nabla}\delta(\boldsymbol{r})||\mathsf{M}^{(2)}\right) \ . \tag{27}$$

This result can be written also in terms of irreducible tensors, but this is only an optional problem, the main objective of expressing the $\delta-$singularities of the field being realized by the result (27). If we introduce the vector $\boldsymbol{N}$ defined by the components

$$N_i = \varepsilon_{ijk}\mathsf{M}_{jk} = \frac{2}{3}\int_{\mathcal{D}} \mathrm{d}^3x \ \left[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{J})\right]_i \ , \tag{28}$$

equation (26) can be written as

$$\mathsf{M}_{ij} = \mathcal{M}_{ij} + \frac{1}{2}\varepsilon_{ijk}N_k \ . \tag{29}$$

The insertion of equation (29) in equation (27) gives

$$\left(\boldsymbol{B}^{(2)}(\boldsymbol{r})\right)_{(0)} = -\frac{3\mu_0}{10}\boldsymbol{\mathcal{M}}^{(2)}||\boldsymbol{\nabla}\delta_\varepsilon(\boldsymbol{r}) - \frac{\mu_0}{4}\boldsymbol{N}\times\boldsymbol{\nabla}\delta_\varepsilon(\boldsymbol{r}) \ . \tag{30}$$

Beginning from $n = 3$, the symmetric projection of the magnetic moment is not the same with the **STF** projection and, consequently, the second step of the procedure becomes an actual calculation. Let us write the regularized expression of the $3-rd$ order magnetic field:

$$\left(\boldsymbol{B}^{(3)}(\boldsymbol{r})\right)_{reg} = \frac{\mu_0}{24\pi}\boldsymbol{\nabla}^4||\frac{\mathsf{M}^{(3)}}{r_\varepsilon} + \frac{\mu_0}{6}\boldsymbol{\nabla}^2\delta_\varepsilon(\boldsymbol{r})||\mathsf{M}^{(3)} \ . \tag{31}$$

Writing the identity

$$\begin{aligned}\mathsf{M}_{ijk} &= \frac{1}{3}\left(\mathsf{M}_{ijk}+\mathsf{M}_{kji}+\mathsf{M}_{ikj}\right) + \frac{1}{3}\left[\left(\mathsf{M}_{ijk}-\mathsf{M}_{kji}\right)+\left(\mathsf{M}_{ijk}-\mathsf{M}_{ikj}\right)\right]\\ &= \overset{\leftrightarrow}{\mathsf{M}}_{ijk} + \frac{1}{3}\left[\left(\mathsf{M}_{ijk}-\mathsf{M}_{kji}\right)+\left(\mathsf{M}_{ijk}-\mathsf{M}_{ikj}\right)\right] \ ,\end{aligned} \tag{32}$$

we introduce the **STF** projection $\boldsymbol{\mathcal{M}}^{(3)}$ of the symmetric tensor $\overset{\leftrightarrow}{\mathsf{M}}^{(3)}$ by the equation

$$\mathcal{M}_{ijk} = \overset{\leftrightarrow}{\mathsf{M}}_{ijk} + \delta_{\{ij}\widetilde{\Lambda}_{k\}}, \quad \widetilde{\Lambda}_i = \frac{1}{5}\overset{\leftrightarrow}{\mathsf{M}}_{qqi} = \frac{1}{15}\mathsf{M}_{qqi} \ . \tag{33}$$

Employing equation (32) and since $\left[\left(\mathsf{M}_{ijk}-\mathsf{M}_{kji}\right)+\left(\mathsf{M}_{ijk}-\mathsf{M}_{ikj}\right)\right]\partial_i\partial_j\partial_k\partial_l(1/r_\varepsilon) = 0$, we can write

$$\boldsymbol{\nabla}^4||\frac{\mathsf{M}^{(3)}}{r_\varepsilon} = \boldsymbol{\nabla}^4||\frac{\overset{\leftrightarrow}{\mathsf{M}}^{(3)}}{r_\varepsilon} \ .$$

Instead of equation (31), we will have

$$\left(\boldsymbol{B}^{(3)}(\boldsymbol{r})\right)_{reg} = \frac{\mu_0}{24\pi}\boldsymbol{\nabla}^4||\frac{\overset{\leftrightarrow}{\mathsf{M}}^{(3)}}{r_\varepsilon} + \frac{\mu_0}{6}\boldsymbol{\nabla}^2\delta_\varepsilon(\boldsymbol{r})||\mathsf{M}^{(3)} \ .$$

The first term from the right-hand side of the above equation can be processed as in the case of $\boldsymbol{E}^{(3)}$, equation (14), employing the result given by equation (21) with $\varepsilon_0 \to 1/\mu_0$ and $\Lambda \to \widetilde{\Lambda}$ such that, finally,

$$\left(\boldsymbol{B}^{(3)}(\boldsymbol{r})\right)_{(0)} = -\frac{\mu_0}{14}\boldsymbol{\mathcal{M}}^{(3)}||\boldsymbol{\nabla}^2\delta(\boldsymbol{r}) - \frac{\mu_0}{2}\widetilde{\Lambda}||\boldsymbol{\nabla}^2\delta(\boldsymbol{r}) + \frac{\mu_0}{6}\boldsymbol{\nabla}^2\delta(\boldsymbol{r})||\mathsf{M}^{(3)} \ . \tag{34}$$

4. General formulas - a mathematical digression

The general expansions of the static electric and magnetic fields are given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n\geq 0}\frac{(-1)^{n-1}}{n!}\mathsf{P}^{(n)}||\boldsymbol{\nabla}^{n+1}\frac{1}{r} \tag{35}$$

and

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n!}\left[\boldsymbol{\nabla}^{n+1}||\frac{\mathsf{M}^{(n)}}{r} - \boldsymbol{\nabla}^{n-1}||\Delta\frac{\mathsf{M}^{(n)}}{r}\right] . \tag{36}$$

The general definition of the electric $n-th$ order moment is given by

$$\mathsf{P}^{(n)} = \int_{\mathcal{D}} \mathrm{d}^3x\; \boldsymbol{r}^n\,\rho(\boldsymbol{r}) : \quad \mathsf{P}_{i_1\ldots i_n} = \int_{\mathcal{D}} \mathrm{d}^3x\, x_{i_1}\ldots x_{i_n}\,\rho(\boldsymbol{r}) ,$$

and for the magnetic $n-th$ order moment by [3]

$$\mathsf{M}^{(n)} = \frac{n}{n+1}\int_{\mathcal{D}} \mathrm{d}^3x\,\boldsymbol{r}^n\times\boldsymbol{J} : \quad \mathsf{M}_{i_1\ldots i_n} = \int_{\mathcal{D}} \mathrm{d}^3x\, x_{i_1}\ldots x_{i_{n-1}}\,(\boldsymbol{r}\times\boldsymbol{J}) .$$

In Ref. [6], a formula for expressing the ordinary $n-th$ order derivative of $1/r^m$ is given (equation (4.16) from this reference). With our notation,

$$\partial_{i_1}\ldots\partial_{i_n}\frac{1}{r^m} = \sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^{n-k}m(m+2)\ldots(2n-2k+m-2)}{r^{2n-2k+m}} \times\; \delta_{\{i_1 i_2}\ldots\delta_{i_{2k-1}\,i_{2k}}x_{2k+1}\ldots x_{i_n\}} , \tag{37}$$

where $[\alpha]$ is the integer part of α. This formula can be employed for calculating the derivatives of $1/r_\varepsilon$ by the simple substitution $r\rightarrow r_\varepsilon$ in equation (37). We are interested of the case m=1:

$$\partial_{i_1}\ldots\partial_{i_n}\frac{1}{r_\varepsilon} = \sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^{n-k}(2n-2k-1)!!}{r_\varepsilon^{2n-2k+1}}\delta_{\{i_1 i_2}\ldots\delta_{i_{2k-1}i_{2k}}x_{i_{2k+1}}\ldots x_{i_n\}} . \tag{38}$$

We also need the formula for the derivatives of $\delta_\varepsilon(\boldsymbol{r})$:

$$\partial_{i_1}\ldots\partial_{i_n}\delta_\varepsilon(\boldsymbol{r}) = \frac{\varepsilon^2}{4\pi}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^{n-k}(2n-2k+3)!!}{r_\varepsilon^{2n-2k+5}}\delta_{\{i_1 i_2}\ldots\delta_{i_{2k-1}i_{2k}}x_{i_{2k+1}}\ldots x_{i_n\}} . \tag{39}$$

Equations (38) and (39) will be employed in the following for expressing the singular terms with point-like support of the electric and magnetic fields for arbitrary n.
Let us firstly take the regularized expression of the $n-th$ order multipole electric field

$$\left(\boldsymbol{E}^{(n)}(\boldsymbol{r})\right)_{reg} = \frac{(-1)^{n-1}}{4\pi\varepsilon_0\,n!}\mathsf{P}^{(n)}||\boldsymbol{\nabla}^{n+1}\frac{1}{r_\varepsilon} = \frac{(-1)^{n-1}}{4\pi\varepsilon_0\,n!}\,\boldsymbol{e}_i\,\mathsf{P}_{i_1\ldots i_n}\,\partial_i\,\partial_{i_1}\ldots\partial_{i_n}\frac{1}{r_\varepsilon} . \tag{40}$$

The **STF** projection of the tensor $\mathsf{P}^{(n)}$ is realized by an obvious generalization of equation (15):

$$\mathsf{P}_{i_1\ldots i_n} = \mathcal{P}_{i_1\ldots i_n} + \delta_{\{i_1 i_2}\,\Lambda_{i_3\ldots i_n\}} , \tag{41}$$

where $\Lambda^{(n-2)}$ is a totally symmetric tensor. This last tensor has a general expression given in Refs. [7] and [8] (the detracer theorem) which, with our notation, is written as

$$\Lambda_{i_1\ldots i_{n-2}}[\mathsf{P}^{(n)}] = \sum_{m=0}^{[n/2-1]} \frac{(-1)^m[2n-1-2(m+1)]!!}{(m+1)(2n-1)!!}\delta_{\{i_1 i_2} \ldots \delta_{i_{2m-1} i_{2m}} \mathsf{P}^{(n:\,m+1)}_{i_{2m+1}\ldots i_{n-2}\}} \ .$$

$\mathsf{P}^{(n:m)}_{i_{2m+1}\ldots i_n}$ denotes the components of the $(n-2\,m)-th$ order tensor obtained from $\mathbf{P}^{(n)}$ by contracting m pairs of symbols i. This theorem, though not employed in the present paper, is given for the reader interested in extending these calculation for higher orders. Further, we insert formula (41) in equation (40) and we focus on the calculation of the contraction of the electric moment tensor with the derivative one:

$$\mathbf{P}^{(n)}||\boldsymbol{\nabla}^{n+1}\frac{1}{r_\varepsilon} = \boldsymbol{e}_i\, \mathsf{P}_{i_1\ldots i_n}\, \partial_i\, \partial_{i_1} \ldots \partial_{i_n} \frac{1}{r_\varepsilon} = \boldsymbol{\mathcal{P}}^{(n)}||\boldsymbol{\nabla}^{n+1}\frac{1}{r_\varepsilon}$$
$$+\, \boldsymbol{e}_i\, \partial_i\, \delta_{\{i_1 i_2}\, \Lambda_{i_3} \ldots i_{n\}} \partial_{i_1} \ldots \partial_{i_n}\frac{1}{r_\varepsilon} = \boldsymbol{\mathcal{P}}^{(n)}||\boldsymbol{\nabla}^{n+1}\frac{1}{r_\varepsilon} + \frac{n(n-1)}{2}\boldsymbol{e}_i \partial_i\, \partial_{i_1} \ldots \partial_{i_{n-2}} \Delta \frac{1}{r_\varepsilon}\ ,$$

where the symmetry of the tensors $\mathbf{\Lambda}^{(n-2)}$ and $\boldsymbol{\nabla}^n$ is considered. It is realized a first separation of $\delta-$singularities set by the introduction of the δ_ε function:

$$\mathbf{P}^{(n)}||\boldsymbol{\nabla}^{n+1}\frac{1}{r_\varepsilon} = \boldsymbol{\mathcal{P}}^{(n)}||\boldsymbol{\nabla}^{n+1}\frac{1}{r_\varepsilon} - 4\pi\frac{n(n-1)}{2}\boldsymbol{e}_i\, \partial_i\, \partial_{i_1} \ldots \partial_{i_{n-2}}\, \delta_\varepsilon(\boldsymbol{r})$$
$$= \boldsymbol{\mathcal{P}}^{(n)}||\boldsymbol{\nabla}^{n+1}\frac{1}{r_\varepsilon} - 4\pi\frac{n(n-1)}{2}\, \mathbf{\Lambda}^{(n-2)}||\boldsymbol{\nabla}^{n-1}\delta_\varepsilon(\boldsymbol{r})\ . \tag{42}$$

Employing the formula (38), we can write

$$\boldsymbol{\mathcal{P}}^{(n)}||\boldsymbol{\nabla}^{n+1}\frac{1}{r_\varepsilon} = -\boldsymbol{e}_{i_{n+1}}\, \mathcal{P}_{i_1\ldots i_n} \partial_{i_1} \ldots \partial_{i_{n+1}}\frac{1}{r_\varepsilon}$$
$$= -\boldsymbol{e}_{i_{n+1}}\, \mathcal{P}_{i_1\ldots i_n} \sum_{k=0}^{[(n+1)/2]} \frac{(-1)^{n-k}(2n-2k+1)!!}{r_\varepsilon^{2n-2k+3}}\delta_{\{i_1 i_2} \ldots \delta_{i_{2k-1} i_{2k}}\, x_{i_{2k+1}} \ldots x_{i_{n+1}\}}$$
$$= -\boldsymbol{e}_{i_{n+1}}\, \mathcal{P}_{i_1\ldots i_n}\left[\frac{(-1)^n(2n+1)!!\, x_{i_1} \ldots x_{i_{n+1}}}{r_\varepsilon^{2n+3}} - \frac{(-1)^n(2n-1)!!\ \delta_{\{i_1 i_2}\, x_{i_3} \ldots x_{i_{n+1}\}}}{r_\varepsilon^{2n+1}}\right]\ ,$$

since the contractions

$$\mathcal{P}_{i_1\ldots i_n}\, \delta_{\{i_1 i_2} \ldots \delta_{i_{2k-1}} \delta_{i_{2k}}\, x_{i_3} \ldots x_{i_{n+1}}\}$$

vanish for $k \geq 2$. We perform the separation of the singular part writing:

$$\boldsymbol{\mathcal{P}}^{(n)}||\boldsymbol{\nabla}^{n+1}\frac{1}{r_\varepsilon} = (-1)^{n-1}\boldsymbol{e}_{i_{n+1}}\mathcal{P}_{i_1\ldots i_n}$$
$$\times\left[\frac{(2n+1)!!\, x_{i_1} \ldots i_{n+1} - (2n-1)!!\, r^2\, \delta_{\{i_1 i_2}\, x_{i_3} \ldots x_{i_{n+1}\}}}{r_\varepsilon^{2n+3}}\right.$$
$$\left. - \frac{(2n-1)!!\, \varepsilon^2 \delta_{\{i_1 i_2} x_{i_3} \ldots x_{i_{n+1}\}}}{r_\varepsilon^{2n+3}}\right]\ . \tag{43}$$

Equation (39) gives the following relation:

$$-\frac{\varepsilon^2\, x_{i_1} \ldots x_{i_{n-1}}}{r_\varepsilon^{2n+3}} = -\frac{4\pi\, (-1)^n}{(2n+1)!!}\, \partial_{i_1} \ldots \partial_{i_{n-1}}\, \delta_\varepsilon(\boldsymbol{r}) \tag{44}$$
$$- \frac{(-1)^n}{(2n+1)!!}\sum_{k=1}^{[(n-1)/2]} \frac{(2n-2k+1)!!}{r_\varepsilon^{2n-2k+3}}\, \delta_{\{i_1 i_2} \ldots \delta_{i_{2k-1} i_{2k}} x_{i_{2k+1}} \ldots x_{i_{n-1}\}}\ .$$

Inserting equation (44) in equation (43), we obtain

$$\begin{aligned}
&\boldsymbol{\mathcal{P}}^{(n)}||\boldsymbol{\nabla}^{n+1}\frac{1}{r_\varepsilon} \\
&= (-1)^{n-1}\boldsymbol{e}_{i_{n+1}}\mathcal{P}_{i_1\ldots i_n}\left[\frac{(2n+1)!!\,x_{i_1}\ldots i_{n+1}-(2n-1)!!\,r^2\,\delta_{\{i_1 i_2}\,x_{i_3}\ldots x_{i_{n+1}\}}}{r_\varepsilon^{2n+3}}\right] \\
&\quad+\sum_{k\geq 1}(\ldots)-\frac{4\pi}{2n+1}\,\boldsymbol{e}_{i_{n+1}}\mathcal{P}_{i_1\ldots i_n}\delta_{\{i_1 i_2}\,\partial_{i_3}\ldots\partial_{i_{n+1}\}}\,\delta_\varepsilon(\boldsymbol{r})\ . \qquad (45)
\end{aligned}$$

In the above equation, the symbol $\sum_{k\geq1}(\ldots)$ represents the contraction of $\mathcal{P}_{i_1\ldots i_n}$ with a sum in which each term contains a Kronecker symbols product with at least two factors, one of these factors being δ_{i_j,i_k} with $j,\,k\,\leq n$. Consequently, all these contractions are vanishing. With this result, we can express the $\delta-$singularity from equation (45):

$$\begin{aligned}
\left(\boldsymbol{\mathcal{P}}^{(n)}||\boldsymbol{\nabla}^{n+1}\frac{1}{r}\right)_{(0)} &= -\,\frac{4\pi}{2n+1}\boldsymbol{e}_i\,\mathcal{P}_{i_1\ldots i_n}\,\delta_{\{ii_1}\,\partial_{i_2}\ldots\partial_{i_n\}}\delta(\boldsymbol{r}) \\
&= -\,4\pi\frac{n}{2n+1}\,\boldsymbol{\mathcal{P}}^{(n)}||\boldsymbol{\nabla}^{n-1}\delta(\boldsymbol{r})\ . \qquad (46)
\end{aligned}$$

From equations (40), (42) and (46), we can write the $\delta-$singularity of $\boldsymbol{E}^{(n)}$:

$$\left(\boldsymbol{E}^{(n)}(\boldsymbol{r})\right)_{(0)} = \frac{(-1)^n}{\varepsilon_0\,(n-1)!}\left(\frac{1}{2n+1}\boldsymbol{\mathcal{P}}^{(n)}+\frac{n-1}{2}\boldsymbol{\Lambda}^{(n-2)}\right)||\boldsymbol{\nabla}^{n-1}\,\delta(\boldsymbol{r})\ . \qquad (47)$$

Let us consider finally the singularity of the magnetic field. The tensor $\mathsf{M}^{(n)}$ is symmetric only in the first $n-1$ indices and satisfies the property

$$\mathsf{M}_{i_1\ldots i_{n-2}\,qq}=0\ .$$

In the first step, we must obtain the symmetric projection $\overset{\leftrightarrow}{\mathsf{M}}^{(n)}$ of the tensor $\mathsf{M}^{(n)}$. For $n\geq 3$, we can generalize equation (25) writing the identity:

$$\begin{aligned}
\mathsf{M}_{i_1\ldots i_n} &= \frac{1}{n}\left(\mathsf{M}_{i_1\ldots i_n}+\mathsf{M}_{i_n i_2\ldots i_{n-1}i_1}+\ldots+\mathsf{M}_{i_1\ldots i_n i_{n-1}}\right) \\
&\quad+\frac{1}{n}\left[\left(\mathsf{M}_{i_1\ldots i_n}-\mathsf{M}_{i_n\ldots i_{n-1}i_1}\right)+\ldots\left(\mathsf{M}_{i_1\ldots i_n}-\mathsf{M}_{i_1\ldots i_{n-2}i_n\,i_{n-1}}\right)\right] \\
&= \overset{\leftrightarrow}{\mathsf{M}}_{i_1\ldots i_n}+\frac{1}{n}\left[\left(\mathsf{M}_{i_1\ldots i_n}-\mathsf{M}_{i_n\ldots i_{n-1}i_1}\right)+\ldots\left(\mathsf{M}_{i_1\ldots i_n}-\mathsf{M}_{i_1\ldots i_{n-2}i_n\,i_{n-1}}\right)\right]\ , \qquad (48)
\end{aligned}$$

where $\overset{\leftrightarrow}{\mathsf{M}}^{(n)}$ represents the symmetric part of the tensor $\mathsf{M}^{(n)}$. Let us write the regularized expression of the $n-th$ order multipole magnetic field:

$$\left(\boldsymbol{B}^{(n)}(\boldsymbol{r})\right)_{reg} = \frac{(-1)^{n-1}\mu_0}{4\pi\,n!}\left[\boldsymbol{\nabla}^{n+1}||\frac{\mathsf{M}^{(n)}}{r_\varepsilon}-\boldsymbol{\nabla}^{n-1}||\Delta\frac{\mathsf{M}^{(n)}}{r_\varepsilon}\right]\ . \qquad (49)$$

The insertion of equation (48) in the above equation gives

$$\boldsymbol{\nabla}^{n+1}||\frac{\mathsf{M}^{(n)}}{r_\varepsilon}=\boldsymbol{\nabla}^{n+1}||\frac{\overset{\leftrightarrow}{\mathsf{M}}^{(n)}}{r_\varepsilon}\ , \qquad (50)$$

since

$$\partial_{i_1}\ldots\partial_{i_n}\frac{1}{r_\varepsilon}\left[\left(\mathsf{M}_{i_1\ldots i_n}-\mathsf{M}_{i_n\ldots i_{n-1}i_1}\right)+\ldots\left(\mathsf{M}_{i_1\ldots i_n}-\mathsf{M}_{i_1\ldots i_{n-2}i_n\,i_{n-1}}\right)\right]=0\ .$$

Let us denote

$$\widetilde{\Lambda}^{(n-2)} = \boldsymbol{\Lambda}\left(\overset{\leftrightarrow}{\mathsf{M}}^{(n)}\right) : \ \overset{\leftrightarrow}{\mathsf{M}}_{i_1 \dots i_n} = \mathcal{M}_{i_1 \dots i_n} + \delta_{\{i_1 i_2}\widetilde{\Lambda}_{i_3 \dots i_n\}} \ . \tag{51}$$

With this notation,

$$\begin{aligned}\boldsymbol{\nabla}^{n+1}||\frac{\mathsf{M}^{(n)}}{r_\varepsilon} &= \boldsymbol{\nabla}^{n+1}||\frac{\boldsymbol{\mathcal{M}}^{(n)}}{r_\varepsilon} + \boldsymbol{e}_i\, \partial_i \partial_{i_1} \dots \partial_{i_n}\, \delta_{\{i_1 i_2}\, \widetilde{\Lambda}_{i_3 \dots i_n\}}\, \frac{1}{r_\varepsilon} \\ &= \boldsymbol{\nabla}^{n+1}||\frac{\boldsymbol{\mathcal{M}}^{(n)}}{r_\varepsilon} + \frac{n(n-1)}{2} \boldsymbol{e}_i \widetilde{\Lambda}_{i_1 \dots i_{n-2}} \partial_i\, \partial_{i_1} \dots \partial_{i_{n-2}}\, \Delta \frac{1}{r_\varepsilon} \ ,\end{aligned}$$

i.e.

$$\boldsymbol{\nabla}^{n+1}||\frac{\mathsf{M}^{(n)}}{r_\varepsilon} = \boldsymbol{\nabla}^{n+1}||\frac{\boldsymbol{\mathcal{M}}^{(n)}}{r_\varepsilon} - 4\pi \frac{n(n-1}{2} \widetilde{\boldsymbol{\Lambda}}^{(n-2)}||\boldsymbol{\nabla}^{n-1}\delta_\varepsilon(\boldsymbol{r}) \ . \tag{52}$$

The last term from equation (49) can be written as

$$\boldsymbol{\nabla}^{n-1}||\Delta \frac{\mathsf{M}^{(n)}}{r_\varepsilon} = -4\pi\, \boldsymbol{e}_i\, \partial_{i_1} \dots \partial_{i_{n-1}}\, \delta_\varepsilon(\boldsymbol{r})\, \mathsf{M}_{i_1 \dots i_{n-1}\, i} = -4\pi\, \boldsymbol{\nabla}^{n-1}\delta_\varepsilon(\boldsymbol{r})||\mathsf{M}^{(n)} \ . \tag{53}$$

It remains to separate the singularity of interest in the first term from the right-hand side of equation (52). In this case, we can employ equation (46) with the substitution $\boldsymbol{\mathcal{P}} \to \boldsymbol{\mathcal{M}}$ obtaining

$$\left(\boldsymbol{\nabla}^{n+1}||\frac{\mathsf{M}^{(n)}}{r_\varepsilon}\right)_{(0)} = -4\pi\, \frac{n}{2n+1} \boldsymbol{\mathcal{M}}^{(n)}||\boldsymbol{\nabla}^{n-1}\delta(\boldsymbol{r}) - 4\pi \frac{n(n-1}{2} \widetilde{\boldsymbol{\Lambda}}^{(n-2)}||\boldsymbol{\nabla}^{n-1}\delta(\boldsymbol{r}) \ .$$

With this last result and with equations (49) and (53) we can write

$$\begin{aligned}\left(\boldsymbol{B}^{(n)}(\boldsymbol{r})\right)_{(0)} = \frac{(-1)^n \mu_0}{n!} &\left[\frac{n}{2n+1} \boldsymbol{\mathcal{M}}^{(n)}||\boldsymbol{\nabla}^{n-1}\delta_\varepsilon(\boldsymbol{r}) + \frac{n(n-1}{2} \widetilde{\boldsymbol{\Lambda}}^{(n-2)}||\boldsymbol{\nabla}^{n-1}\delta(\boldsymbol{r})\right. \\ &\left. - \ \boldsymbol{\nabla}^{n-1}\delta(\boldsymbol{r})||\mathsf{M}^{(n)}\right] \ .\end{aligned} \tag{54}$$

With this equation, the separation of the $\delta-$singularities for the magnetic field can be considered finished.

5. Concluding remarks

Along this article, we presented an alternative and, we believe, simpler procedure to determine the $\delta-$singularities of the static electromagnetic field. The method is based on a generalization of an identity involving the regularized derivative of $1/r$ and the regularized function $\delta_\varepsilon(\boldsymbol{r})$. Section 2 presented the discussion for the dipole, quadrupole and octopole of the static electric field and section 3 treated the magnetic case. Section 4 gave the generalization of the results for arbitrary n multipoles.

All the results from a recent paper [9] are recovered in the static case. These results are obtained in Ref. [9] by a generalization of the procedure employed in Ref. [4]. The method presented in the current paper is more direct and avoids the discussions related to the regularization procedures.

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